

XVII. *Researches in Physical Astronomy.* By J. W. LUBBOCK, Esq. V.P. and  
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Read June 7, 1832.

I SUBJOIN some further developments in the Theory of the Moon, which I have thought it advisable to give at length, in order to save the trouble of the calculator and to avoid the danger of mistake, although they may be obtained with great readiness and facility by means of the Table which I have given for the purpose.

While on the one hand it seems desirable to introduce into the science of Physical Astronomy a greater degree of uniformity, by bringing to perfection a Theory of the Moon, founded on the integration of the equations which are used in the planetary theory, it seems also no less important to complete in the latter the method hitherto applied solely to the periodic inequalities. Hitherto those terms in the disturbing function which give rise to the secular inequalities have been detached, and the stability of the system has been inferred by means of the integration of certain equations, which are linear when the higher powers of the eccentricities are neglected, and from considerations founded on the variation of the elliptic constants.

The stability of the system may, I think, also be inferred from the expressions which result at once from the direct integration of the differential equations. In fact, in order that the system may be stable, it is necessary that none of the angles under the sign *sine* or *cosine* be imaginary, which terms would then be converted into exponentials, and be subject to indefinite increase. In the lunar theory, the arbitrary quantities being determined with that view, according to the method here given, the angles which are introduced may be reduced to the difference of the mean motions of the sun and moon, their mean anomalies and the argument of the moon's latitude \*.

\* So that however far the approximation be carried, all the arguments, in the expressions of  $r$ ,  $s$ , and  $\lambda$  are of the form,  $it \pm kx \pm lz \pm my$ ;  $i$ ,  $k$ ,  $l$ , and  $m$  being some whole numbers.

This being the case, no imaginary angles are introduced, if the quantities  $c$  and  $g$  are rational. This theory, which does not seem to be limited by the direction of the moon's motion, and which may be extended without difficulty, already embraces the terms which are included in the secular inequalities, and which are derived from the constant part of  $R$  carried to the order of the squares of the eccentricities. Generally when the method of the variation of constants is employed to determine any inequalities, the development of  $R$  must be carried one degree further, as regards the eccentricities, than the degree which is required of the inequalities sought.

The equation for determining the coefficients of the expression for the reciprocal of the radius vector is,

$$\frac{d^2 \cdot r^2}{2 dt^2} - \frac{d^2 \cdot r^3 \delta \frac{1}{r}}{dt^2} + \frac{3 d^2 \cdot r^4 \left( \delta \frac{1}{r} \right)^2}{2 dt^2} - \frac{2 d^2 \cdot r^5 \left( \delta \frac{1}{r} \right)^3}{dt^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2 \int dR + r \left( \frac{dR}{dr} \right) = 0$$

$$\begin{aligned} r^3 \delta \cdot \frac{1}{r} - \frac{3}{2} \left( r \delta \frac{1}{r} \right)^2 &= \left\{ \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} r_1 - \frac{3e^2}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_3 + r_4) \right. \\ &\quad \left. - \frac{3}{2} \left\{ 2r_0 r_1 + e^2 (r_3 + r_4) r_2 + e_1^2 (r_6 + r_7) r_5 \right\} \right\} \cos 2t + \&c. \\ &\quad - 3e^2 \{ 2r_1 r_2 + 2r_0 r_3 + 2r_0 r_4 \} \end{aligned}$$

$r_n'$  being the coefficient corresponding to the  $n^{\text{th}}$  argument in the development of  $r \delta \frac{1}{r}$ . The development of  $r^3 \delta \frac{1}{r}$  is easily deduced from that of  $r \delta \frac{1}{r}$  given in the Phil. Trans. 1832, Part I. p. 3, and that of  $\left( r \delta \frac{1}{r} \right)^2$  from that of  $\left( \delta \frac{1}{r} \right)^2$ , p. 4. If  $r_n$  is that part of the coefficient of the  $n^{\text{th}}$  argument in the development of the quantity  $r^3 \delta \frac{1}{r} - \frac{3}{2} \left( r \delta \frac{1}{r} \right)^2$  which is independent of  $r_n$ , with a contrary sign;

$$\begin{aligned} r_1 &= \frac{3e^2}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_3 + r_4) + \frac{3}{2} \left\{ 2r_0 r_1 + e^2 (r_3 + r_4) r_2 + e_1^2 (r_6 + r_7) r_5 \right\} \\ &\quad - 3e^2 \{ 2r_1 r_2 + 2r_0 r_3 + 2r_0 r_4 \} \\ r_2 &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (2r_0 + e^2 r_3) + \frac{3}{2} \left\{ (r_4 + r_3) r_1 + 2r_0 r_2 \right\} \\ &\quad - 6 \left\{ r_0^2 + \frac{r_1^2}{2} + \frac{e^2 r_3^2}{2} + \&c. \right\} \end{aligned}$$

$$\begin{aligned} r_3 = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_9 + r_1) + \frac{3}{2} \{ r_1 r_2 + 2 r_0 r_3 \} \\ & - 3 \{ 2 r_0 r_1 + e^2 (r_3 + r_4) r_2 + e^2 (r_6 + r_7) r_5 \} \end{aligned}$$

$$\begin{aligned} r_4 = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_1 + e^2 r_{10}) + \frac{3}{2} \{ r_1 r_2 + 2 r_0 r_4 \} \\ & - 3 \{ 2 r_0 r_1 + e^2 (r_3 + r_4) r_2 + e^2 (r_6 + r_7) r_5 \} \end{aligned}$$

$$r_5 = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{14} + e^2 r_{11}) + \frac{3}{2} \{ r_1 r_7 + r_1 r_6 + 2 r_0 r_5 \}$$

$$r_6 = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{12} + e^2 r_{16}) + \frac{3}{2} \{ r_5 r_1 + 2 r_0 r_6 \}$$

$$r_7 = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{15} + e^2 r_{13}) + \frac{3}{2} \{ r_5 r_1 + 2 r_0 r_7 \}$$

$$\begin{aligned} r_8 = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (2 r_0 + r_2 + e^2 r_{20}) + \frac{e^2 r_2}{16} + \frac{3}{2} \{ r_2^2 + r_4 r_3 + r_1 r_9 + r_1 r_{10} \} \\ & - 3 \{ 2 r_0^2 + r_1^2 + (r_4 + r_3) r_1 + 2 r_0 r_2 \} + 3 \left\{ r_0^2 + \frac{r_1^2}{2} \right\} \end{aligned}$$

$$\begin{aligned} r_9 = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{21} + r_3) + \frac{e^2}{16} r_4 + \frac{3}{2} \{ r_2 r_3 + 2 r_0 r_9 \} \\ & - 3 \{ r_1 r_2 + 2 r_0 r_3 \} + \frac{3}{2} \{ 2 r_0 r_1 + e^2 r_3 r_2 \} \end{aligned}$$

$$\begin{aligned} r_{10} = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_4 + e^2 r_{22}) + \frac{e^2}{16} r_3 + \frac{3}{2} \{ r_4 r_2 + 2 r_0 r_{10} \} \\ & - 3 \{ r_1 r_2 + 2 r_0 r_4 \} + \frac{3}{2} \{ 2 r_0 r_1 + e^2 r_3 r_2 \} \end{aligned}$$

$$\begin{aligned} r_{11} = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_5 + e^2 r_{23}) + \frac{3}{2} \{ r_1 r_{13} + r_1 r_{12} + r_2 r_5 + r_6 r_4 + r_3 r_7 + 2 r_0 r_{11} \} \\ & - 3 \{ r_1 r_7 + r_1 r_6 + 2 r_0 r_5 \} \end{aligned}$$

$$r_{12} = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{24} + r_6) + \frac{3}{2} \{ r_{11} r_1 + r_2 r_6 + r_5 r_3 + 2 r_0 r_{12} \} - 3 \{ r_5 r_1 + 2 r_0 r_6 \}$$

$$r_{13} = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_7 + e^2 r_{25}) + \frac{3}{2} \{ r_{11} r_1 + r_2 r_7 + r_5 r_4 + 2 r_0 r_{13} \} - 3 \{ r_5 r_1 + 2 r_0 r_7 \}$$

$$\begin{aligned} r_{14} = & \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{26} + r_5) + \frac{3}{2} \{ r_{16} r_1 + r_{15} r_1 + r_2 r_5 + r_6 r_3 + r_7 r_4 + 2 r_0 r_{14} \} \\ & - 3 \{ r_1 r_7 + r_1 r_6 + 2 r_0 r_5 \} \end{aligned}$$

$$r_{15} = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{27} + r_7) + \frac{3}{2} \{ r_{14} r_1 + r_2 r_7 + r_5 r_3 \} - 3 \{ r_5 r_1 + 2 r_0 r_7 \}$$

$$r_{16} = \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (r_6 + e^2 r_{28}) + \frac{3}{2} \{ r_{14} r_1 + r_2 r_6 + r_5 r_4 \} - 3 \{ r_5 r_1 + 2 r_0 r_6 \}$$

$$\begin{aligned}
r_{17} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{32} + e^2 r_{29}) + \frac{3}{2} \left\{ r_5^2 + r_7 r_6 + r_1 r_{18} + r_1 r_{19} \right\} \\
r_{18} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{30} + e^2 r_{34}) + \frac{3}{2} \left\{ r_{17} r_1 + r_5 r_6 \right\} \\
r_{19} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) (e^2 r_{33} + e^2 r_{31}) + \frac{3}{2} \left\{ r_{17} r_1 + r_7 r_5 \right\} \\
r_{20} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_8 + \frac{1}{8} r_0 & r_{21} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_9 + \frac{1}{16} r_1 \\
r_{22} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{10} + \frac{1}{16} r_1 & r_{23} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{11} \\
r_{24} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{12} & r_{25} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{13} \\
r_{26} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{14} & r_{27} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{15} & r_{28} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{16} \\
r_{29} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{17} & r_{30} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{18} & r_{31} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{19} \\
r_{32} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{17} & r_{33} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{19} & r_{34} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{18} \\
r_{35} &= 0 & r_{36} &= 0 & r_{37} &= 0 \\
r_{38} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{20} + \frac{1}{16} r_2 & r_{39} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{21} + \frac{1}{16} r_3 \\
r_{40} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{22} + \frac{1}{16} r_4 & r_{41} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{23} + \frac{1}{16} r_5 \\
r_{42} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{24} + \frac{1}{16} r_6 & r_{43} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{25} + \frac{1}{16} r_7 \\
r_{44} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{26} + \frac{1}{16} r_5 & r_{45} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{27} + \frac{1}{16} r_7 \\
r_{46} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{28} + \frac{1}{16} r_6 & r_{47} &= \frac{3}{2} \left( 1 + \frac{3}{8} e^2 \right) r_{29}
\end{aligned}$$

Let  $R_n$  be the coefficient corresponding to the  $n^{\text{th}}$  argument in the development of  $aR + a\delta R$ ,  $mR'_n$  the coefficient corresponding to the  $n^{\text{th}}$  argument in the development of  $a\delta dR$  with its sign changed, Phil. Trans. 1832, p. 161, so that, for example, when the square of the disturbing force is neglected,

$$R_1 = -\frac{3}{4} \frac{m_1}{\mu} \frac{a^3}{a_1^3} \text{ then}$$

$$r_1 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-2m)^2}{(2-2m)^2-1} r_1 - \frac{2}{(2-2m)^2-1} \left\{ \left\{ \frac{2}{2-2m} + 1 \right\} R_1 + \frac{m}{2-2m} R'_1 \right\}$$

$$c^2 \left\{ 1 - \frac{e^2}{8} - r_2 \right\} = 1 - \frac{e^2}{8} - 2 \left\{ \left\{ \frac{1}{c} + 1 \right\} R_2 + \frac{m}{c} R'_2 \right\}$$

$$r_3 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-2m-c)^2}{(2-2m-c)^2-1} r_3 - \frac{2}{(2-2m-c)^2-1} \left\{ \left\{ \frac{2-c}{2-2m-c} + 1 \right\} R_3 + \frac{m}{2-2m-c} R_3' \right\}$$

$$r_4 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-2m+c)^2}{(2-2m+c)^2-1} r_4 - \frac{2}{(2-2m+c)^2-1} \left\{ \left\{ \frac{2+c}{2-2m+c} + 1 \right\} R_4 + \frac{m}{2-2m+c} R_4' \right\}$$

$$r_5 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{m^2}{m^2-1} r_5 - \frac{2}{m^2-1} \{ R_5 + R_5' \}$$

$$r_6 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-3m)^2}{(2-3m)^2-1} r_6 - \frac{2}{(2-3m)^2-1} \left\{ \left\{ \frac{2}{2-3m} + 1 \right\} R_6 + \frac{m}{2-3m} R_6' \right\}$$

$$r_7 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-m)^2}{(2-m)^2-1} r_7 - \frac{2}{(2-m)^2-1} \left\{ \left\{ \frac{2}{2-m} + 1 \right\} R_7 + \frac{m}{2-m} R_7' \right\}$$

$$c^2 \left\{ 1 - \frac{e^2}{3} - 2r_8 - 2r_8' \right\} = 1 - \frac{e^2}{3} - 2 \left\{ \left\{ \frac{1}{c} + 1 \right\} R_8 + \frac{m}{2c} R_8' \right\}$$

$$r_9 \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-2m-2c)^2}{(2-2m-2c)^2-1} r_9 - \frac{2}{(2-2m-2c)^2-1} \left\{ \left\{ \frac{2-2c}{2-2m-2c} + 1 \right\} R_9 + \frac{m}{2-2m-2c} R_9' \right\}$$

$$r_{10} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-2m+2c)^2}{(2-2m+2c)^2-1} r_{10} - \frac{2}{(2-2m+2c)^2-1} \left\{ \left\{ \frac{2+2c}{2-2m+2c} + 1 \right\} R_{10} + \frac{m}{2-2m+2c} R_{10}' \right\}$$

$$r_{11} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(c+m)^2}{(c+m)^2-1} r_{11} - \frac{2}{(c+m)^2-1} \left\{ \left\{ \frac{c}{c+m} + 1 \right\} R_{11} + \frac{m}{c+m} R_{11}' \right\}$$

$$r_{12} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-c-3m)^2}{2-c-3m)^2-1} r_{12} - \frac{2}{(2-c-3m)^2-1} \left\{ \left\{ \frac{2-c}{2-c-3m} + 1 \right\} R_{12} + \frac{m}{2-c-3m} R_{12}' \right\}$$

$$r_{13} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-m+c)^2}{(2-m+c)^2-1} r_{13} - \frac{2}{(2-m+c)^2-1} \left\{ \left\{ \frac{2+c}{2-m+c} + 1 \right\} R_{13} + \frac{m}{2-m+c} R_{13}' \right\}$$

$$r_{14} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(c-m)^2}{(c-m)^2-1} r_{14} - \frac{2}{(c-m)^2-1} \left\{ \left\{ \frac{c}{c-m} + 1 \right\} R_{14} + \frac{m}{c-m} R_{14}' \right\}$$

$$r_{15} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-m-c)^2}{(2-m-c)^2-1} r_{15} \\ - \frac{2}{(2-m-c)^2-1} \left\{ \left\{ \frac{2-c}{2-m-c} + 1 \right\} R_{15} + \frac{m}{2-m-c} R_{15}' \right\}$$

$$r_{16} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-3m+c)^2}{(2-3m+c)^2-1} r_{16} \\ - \frac{2}{(2-3m+c)^2-1} \left\{ \left\{ \frac{2+c}{2-3m+c} + 1 \right\} R_{16} + \frac{m}{2-3m+c} R_{16}' \right\}$$

$$r_{17} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{4m^2}{4m^2-1} r_{17} - \frac{2}{4m^2-1} \left\{ R_{17} + R_{17}' \right\}$$

$$r_{18} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(2-4m)^2}{(2-4m)^2-1} r_{18} \\ - \frac{2}{(2-4m)^2-1} \left\{ \left\{ \frac{2}{2-4m} + 1 \right\} R_{18} + \frac{m'}{2-4m} R_{18}' \right\}$$

$$r_{19} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{4}{3} r_{19} - \frac{2}{3} \left\{ 2R_{19} + \frac{m}{2} R_{19}' \right\}$$

$$r_{101} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(1-m)^2}{(1-m)^2-1} r_{101} \\ - \frac{1}{(1-m)^2-1} \left\{ \left\{ \frac{2}{1-m} + 3 \right\} R_{101} + \frac{2m}{1-m} R_{101}' \right\}$$

$$r_{102} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(1-m-c)^2}{(1-m-c)^2-1} r_{102} \\ - \frac{1}{(1-m-c)^2-1} \left\{ \left\{ \frac{2(1-c)}{1-m-c} + 3 \right\} R_{102} + \frac{2m}{1-m-c} R_{102}' \right\}$$

$$r_{103} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(1-m+c)^2}{(1-m+c)^2-1} r_{103} \\ - \frac{1}{(1-m+c)^2-1} \left\{ \left\{ \frac{2(1+c)}{1-m+c} + 3 \right\} R_{103} + \frac{2m}{1-m+c} R_{103}' \right\}$$

$$r_{104} \left\{ 1 + 3e^2 \left( 1 + \frac{e^2}{8} \right) \right\} = \frac{(1-2m)^2}{(1-2m)^2-1} r_{104} \\ - \frac{1}{(1-2m)^2-1} \left\{ \left\{ \frac{2}{1-2m} + 3 \right\} R_{104} + \frac{2m}{1-2m} R_{104}' \right\}$$

$$m = \cdot 0748013$$

$$c = \cdot 991548$$

$$e = \cdot 0548442$$

Substituting in the preceding equations, and writing the logarithms of the coefficients instead of the coefficients themselves, we get

$$r_1 = 0\cdot 1460995 r_1 - 0\cdot 2308405 R_1 - 8\cdot 5192440 R_1'$$

$$r_3 = -0\cdot 4450058 r_3 + 1\cdot 2154967 R_3 + 9\cdot 8181930 R_3'$$

$$\begin{aligned}
r_4 &= 0.0535010 \mathbf{r}_4 - 9.7596140 R_4 - 7.8675954 R_4' \\
r_5 &= -7.7463524 \mathbf{r}_5 + 0.2995642 R_5 + 0.2995642 R_5' \\
r_6 &= 0.1617938 \mathbf{r}_6 - 0.2917755 R_6 - 8.5887003 R_6' \\
r_7 &= 0.1326574 \mathbf{r}_7 - 0.1741219 R_7 - 8.4541703 R_7' \\
r_9 &= -8.2495414 \mathbf{r}_9 + 0.2456727 R_9 - 0.0558873 R_9' \\
r_{10} &= 0.0267023 \mathbf{r}_{10} - 9.4699640 R_{10} - 7.4508570 R_{10}' \\
r_{11} &= 0.9148582 \mathbf{r}_{11} - 1.4456131 R_{11} - 0.0060992 R_{11}' \\
r_{12} &= 0.1990183 \mathbf{r}_{12} + 1.0704790 R_{12} + 9.6909293 R_{12}' \\
r_{13} &= 0.0504044 \mathbf{r}_{13} - 9.7282013 R_{13} - 7.8306471 R_{13}' \\
r_{14} &= 0.7176313 \mathbf{r}_{14} + 1.4125573 R_{14} + 0.0058216 R_{14}' \\
r_{15} &= -0.8282531 \mathbf{r}_{15} + 1.5070002 R_{15} + 0.0926384 R_{15}' \\
r_{16} &= 0.0568761 \mathbf{r}_{16} - 9.7921334 R_{16} - 7.9057198 R_{16}' \\
r_{17} &= -8.3558051 \mathbf{r}_{17} + 0.3069571 R_{17} + 0.3069571 R_{17}' \\
r_{18} &= 0.1803182 \mathbf{r}_{18} - 0.3576881 R_{18} - 8.6633026 R_{18}' \\
r_{19} &= 0.1210357 \mathbf{r}_{19} - 0.1210357 R_{19} - 8.3928848 R_{19}' \\
r_{101} &= -0.7701834 \mathbf{r}_{101} + 1.5505062 R_{101} + 0.0464175 R_{101}' \\
r_{102} &= -7.6416818 \mathbf{r}_{102} + 0.4365911 R_{102} - 0.3511177 R_{102}' \\
r_{103} &= 0.1340779 \mathbf{r}_{103} - 0.2746455 R_{103} - 8.4613229 R_{103}' \\
r_{104} &= -0.4131392 \mathbf{r}_{104} + 1.2823979 R_{104} + 9.7992116 R_{104}'
\end{aligned}$$

These quantities introduce into the expression for the longitude expressed in sexagesimal seconds, the terms,

$$\begin{aligned}
&+ \{5.4942896 \mathbf{r}_1 - 5.5790306 R_1 - 3.8674341 R_1'\} \sin 2t & [4.7798951] \\
&+ \{-4.8656743 \mathbf{r}_3 + 5.6361652 R_3 + 4.2382615 R_3'\} \sin (2t - x) & [4.1857212] \\
&+ \{3.9544710 \mathbf{r}_4 - 3.6605840 R_4 - 1.7685654 R_4'\} \sin (2t + x) & [3.1463242] \\
&+ \{-2.7130189 \mathbf{r}_5 + 5.2662307 R_5 + 5.2662307 R_5'\} \sin z & [5.7917274] \\
&+ \{3.7530252 \mathbf{r}_6 - 3.8830069 R_6 - 2.1799317 R_6'\} \sin (2t - z) & [3.0408572] \\
&+ \{3.6887576 \mathbf{r}_7 - 3.7302221 R_7 - 2.0102705 R_7'\} \sin (2t + z) & [2.9705948] \\
&+ \{2.2203935 \mathbf{r}_9 - 4.2165248 R_9 + 4.0267394 R_9'\} \sin (2t - 2x) & [4.5469577] \\
&+ \{2.5368240 \mathbf{r}_{10} - 1.9800857 R_{10} - 9.9609787 R_{10}'\} \sin (2t + 2x) & [1.6254969] \\
&+ \{3.4666708 \mathbf{r}_{11} - 3.9974257 R_{11} - 2.5579118 R_{11}'\} \sin (x + z) & [2.2228889]
\end{aligned}$$

$$\begin{aligned}
& + \{ -2.8843819 r_{12} + 3.7558426 R_{12} + 2.3762929 R_{12}' \} \sin (2t - x - z) & [2.4899904] \\
& + \{ 2.1652119 r_{13} - 1.8430088 R_{13} - 9.9454546 R_{13}' \} \sin (2t + x + z) & [1.3488787] \\
& + \{ -3.3350850 r_{14} + 4.0300110 R_{14} + 2.6232753 R_{14}' \} \sin (x - z) & [2.3541741] \\
& + \{ -3.4377718 r_{15} + 4.1169189 R_{15} + 2.7021571 R_{15}' \} \sin (2t - x + z) & [2.3383041] \\
& + \{ 2.1945476 r_{16} - 1.9298049 R_{16} - 0.0433913 R_{16}' \} \sin (2t + x - z) & [1.3946097] \\
& + \{ -1.2465621 r_{17} + 3.1977141 R_{17} + 3.1977141 R_{17}' \} \sin 2z & [3.4147879] \\
& + \{ 2.0153626 r_{18} - 2.1927325 R_{18} - 0.4983470 R_{18}' \} \sin (2t - 2z) & [1.3033627] \\
& + \{ 1.8857018 r_{19} - 1.8857018 R_{19} - 0.1575509 R_{19}' \} \sin (2t + 2z) & [1.1626061] \\
& + \{ -6.4194035 r_{101} + 7.1997263 R_{101} + 5.6956376 R_{101}' \} \sin t & [5.3481901] \\
& + \{ 3.1744332 r_{102} - 5.9693425 R_{102} + 5.8838691 R_{102}' \} \sin (t - x) & [6.4098870] \\
& + \{ 4.2060990 r_{103} - 4.3466666 R_{103} - 2.5333440 R_{103}' \} \sin (t + x) & [3.4884264] \\
& + \{ -4.3240929 r_{104} + 5.1933516 R_{104} + 3.7101653 R_{104}' \} \sin (t - z) & [3.6803018]
\end{aligned}$$

The preceding expressions serve to show the extent to which the approximation must be carried in the calculation of the quantities  $r$ ,  $R$ , &c.

If we take the term  $5.6361652 R_3$ , since  $\log. \frac{m_1 a^3}{\mu a_i^3} = 7.7464329$ , it is evident that in order not to neglect  $\cdot 01''$  in the value of  $\lambda$ , the coefficient of  $\frac{m_1 a^2}{\mu a_i^3} \cos (2t - x)$  in the development of  $\delta R$  must be calculated exactly to the fifth place of decimals, but not beyond. The number  $4.1857212$  is the logarithm of the quantity  $\frac{e}{(2 - m - c)^2}$ , expressed in sexagesimal seconds, and serves to show in like manner how far the approximation must be carried in the calculation of  $\frac{dR}{d\lambda}$ .

When the square of the disturbing force is neglected,

$$\begin{aligned}
R_2 &= \frac{m_1 a^3}{2 \mu a_i^3} & R_8 &= \frac{m_1 a^3}{8 \mu a_i^3} & r_0 &= -\frac{m_1 a^3}{2 \mu a_i^3} & r_2 &= 3 r_0 & r_8 &= 3 r_0 & r_2 &= 0 \\
c^2 &= 1 + 3 r_0 - \frac{2 m_1 a^3}{\mu a_i^3} = 1 - \frac{7 m_1 a^3}{2 \mu a_i^3}
\end{aligned}$$

The equation of p. 5, line 8, gives  $r_8 = 0$ .

$$\frac{d\lambda}{dt} = \frac{h}{r^2} \left\{ 1 - \frac{1}{h} \int \frac{dR}{d\lambda} dt + \frac{1}{2h^2} \left\{ \int \frac{dR}{d\lambda} dt \right\}^2 \right.$$

$$\left. \frac{h}{r^2} = \frac{h(1 + s^2)}{r^2} = \frac{h}{a^2} \left\{ \frac{a}{r} + a \delta \frac{1}{r} \right\}^2 \left\{ 1 + s^2 \right\} \right.$$



$$= \frac{h}{a^2} \left\{ \frac{a^2}{r^2} + \frac{2a^2}{r} \delta \frac{1}{r} + a^2 \left( \delta \cdot \frac{1}{r} \right)^2 \right\} \left\{ 1 + s^2 \right\}$$

$$s = \gamma \sin y + \gamma s_{147} \sin (2t - y) \text{ nearly}$$

[146]

[147]

$$s^2 = \frac{\gamma^2}{2} + \frac{\gamma^2 s_{147}^2}{2} - \gamma^2 s_{147} \cos 2t - \frac{\gamma^2}{2} \cos 2y + \gamma^2 s_{147} \cos (2t - 2y)$$

(1)

(62)

(63)

$$1 + s^2 = 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \left\{ 1 - \gamma^2 s_{147} \cos 2t - \frac{\gamma^2}{2} \cos 2y + \gamma^2 s_{147} \cos (2t - 2y) \right\}$$

[1]

[62]

[63]

nearly

$$\frac{a^2}{r^2} = 1 + \frac{e^2}{2} \left( 1 + \frac{3}{4} e^2 \right) + 2e \left( 1 + \frac{3}{8} e^2 \right) \cos x + \frac{5}{2} e^2 \left( 1 + \frac{2}{15} e^2 \right) \cos 2x$$

[2]

[8]

$$+ \frac{13}{4} e^3 \cos 3x + \frac{103}{24} e^4 \cos 4x$$

[20]

[38]

$$\frac{a}{r} = 1 + e \left( 1 - \frac{e^2}{8} \right) \cos x + e^2 \left( 1 - \frac{e^2}{3} \right) \cos 2x + \frac{9}{8} e^3 \cos 3x + \frac{4}{3} e^4 \cos 4x$$

[2]

[8]

[20]

[38]

If the coefficients corresponding to the different arguments in the quantity  $\frac{a^2}{r^2}$  be called  $2r_n$  and the coefficients of the different arguments in the development of the quantity

$-n a \left\{ \int \frac{dR}{d\lambda} dt - \frac{1}{2h^2} \left\{ \int \frac{dR}{d\lambda} dt^2 \right\}^2 \right\}$  be called  $\mathfrak{R}_n$ , then

$$2r_0 = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ 1 + \frac{e^2}{2} \left( 1 + \frac{3}{4} e^2 \right) + 2r_0 + r_0^2 + \frac{r_1^2}{2} + \frac{e^2 r_2^2}{2} + \frac{e^2 r_3^2}{2} \frac{e^2 r_4^2}{2} + \frac{e_1^2 r_5^2}{2} + \frac{e_1^2 r_6^2}{2} + \frac{e_1^2 r_7^2}{2} \right\}$$

$$r_1 = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_1 - \gamma^2 s_{147} + \frac{e^2}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_3 + r_4 \right\} + \frac{e^2}{2} \left\{ r_9 + r_{10} \right\} + 2r_0 r_1 + e^2 (r_3 + r_4) r_2 + e_1^2 (r_6 + r_7) r_5 \right\}$$

$$r_2 = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ 1 + \frac{3}{8} e^2 + r_2 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ 2r_0 + e^2 r_8 \right\} + \frac{e^2}{2} r_2 + (r_4 + r_3) r_1 + 2r_0 r_2 \right\}$$

\*  $(s_{147})^2$  is intended.

$$r_3' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_3 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_9 + r_1 - \gamma^2 s_{147} \right\} + \frac{e^2}{2} r_4 + r_1 r_2 + 2 r_0 r_3 \right\}$$

$$r_4' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_4 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_1 - \gamma^2 s_{147} + e^2 r_{10} \right\} + \frac{e^2}{2} r_3 + r_1 r_2 + 2 r_0 r_4 \right\}$$

$$r_5' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_5 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{14} + e^2 r_{11} \right\} + r_1 r_7 + r_1 r_6 + 2 r_0 r_5 \right\}$$

$$r_6' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_6 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{12} + e^2 r_{16} \right\} + r_5 r_1 + 2 r_0 r_6 \right\}$$

$$r_7' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_7 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{15} + e^2 r_{13} \right\} + r_5 r_1 + 2 r_0 r_7 \right\}$$

$$r_8' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_8 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_2 + e^2 r_{20} \right\} + r_0 + \frac{9}{16} e^2 r_2 \right. \\ \left. + r_2^2 + r_4 r_3 + r_1 r_9 + r_1 r_{10} \right\}$$

$$r_9' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_9 + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{21} + r_3 \right\} + \frac{r_1}{2} - \frac{\gamma^2}{2} s_{147} \right. \\ \left. + \frac{9}{16} e^2 r_4 + r_3 r_3 + 2 r_0 r_9 \right\}$$

$$r_{10}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{10} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_4 + e^2 r_{22} \right\} + \frac{r_1}{2} - \frac{\gamma^2}{2} s_{147} \right. \\ \left. + \frac{9}{16} e^2 r_3 + r_4 r_2 + 2 r_0 r_{10} \right\}$$

$$r_{11}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{11} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_5 + e^2 r_{23} \right\} + \frac{e^2}{2} r_{14} + r_1 r_{13} + r_1 r_{12} + r_2 r_5 \right. \\ \left. + r_6 r_4 + r_3 r_7 + 2 r_0 r_{11} \right\}$$

$$r_{12}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{12} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{24} + r_6 \right\} + \frac{e^2}{2} r_{16} + r_{11} r_1 + r_2 r_6 + r_5 r_3 + 2 r_0 r_{12} \right\}$$

$$r_{13}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{13} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_7 + e^2 r_{25} \right\} + \frac{e^2}{2} r_{15} + r_{11} r_1 + r_2 r_7 + r_5 r_4 + 2 r_0 r_{13} \right\}$$

$$r_{14}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{14} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{26} + r_5 \right\} + \frac{e^2}{2} r_{11} + r_{16} r_1 + r_{15} r_1 + r_2 r_5 + r_6 r_3 \right. \\ \left. + r_7 r_4 + 2 r_0 r_{14} \right\}$$

$$r_{15}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{15} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{27} + r_7 \right\} + \frac{e^2}{2} r_{13} + r_{14} r_1 + r_2 r_7 + r_5 r_3 \right\}$$

$$r_{16}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{16} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ r_6 + e^2 r_{23} \right\} + \frac{e^2}{2} r_{12} + r_{14} r_1 + r_2 r_6 + r_5 r_4 \right\}$$

$$r_{17}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{17} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{32} + e^2 r_{29} \right\} + r_5^2 + r_7 r_6 + r_1 r_{18} + r_1 r_{19} \right\}$$

$$r_{18}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{18} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{30} + e^2 r_{34} \right\} + r_{17} r_1 + r_5 r_6 \right\}$$

$$r_{19}' = \left\{ 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s_{147}^2 \right\} \left\{ r_{19} + \frac{1}{2} \left( 1 - \frac{e^2}{8} \right) \left\{ e^2 r_{33} + e^2 r_{31} \right\} + r_{17} r_1 + r_7 r_5 \right\}$$

$$\lambda' = \left\{ \frac{2h r_0'}{a^2} + 2 r_0' \mathfrak{R}_0 + r_1' \mathfrak{R}_1 + e^2 r_3' \mathfrak{R}_3 + e^2 r_4' \mathfrak{R}_4 + e_i^2 r_5' \mathfrak{R}_5 + e_i^2 r_6' \mathfrak{R}_6 + e_i^2 r_7' \mathfrak{R}_7 \right\} t$$

$$+ \frac{1}{2-2m} \{ 2 r_1' + 2 r_0' \mathfrak{R}_1 + 2 r_1' \mathfrak{R}_0 + e^2 r_2' \mathfrak{R}_3 + e^2 r_2' \mathfrak{R}_4 + e^2 r_3' \mathfrak{R}_2 + e^2 r_4' \mathfrak{R}_2$$

$$+ e_i^2 r_5' \mathfrak{R}_6 + e_i^2 r_5' \mathfrak{R}_7 + e_i^2 r_6' \mathfrak{R}_5 + e_i^2 r_7' \mathfrak{R}_5 \} \sin 2t$$

[1]

$$+ \frac{1}{c} \{ 2 r_2' + 2 r_0' \mathfrak{R}_2 + 2 r_2' \mathfrak{R}_0 + r_1' \mathfrak{R}_4 + r_1' \mathfrak{R}_3 + r_3' \mathfrak{R}_1 + r_4' \mathfrak{R}_1 \} e \sin x$$

[2]

$$+ \frac{1}{(2-2m-c)} \{ 2 r_3' + 2 r_0' \mathfrak{R}_3 + 2 r_3' \mathfrak{R}_0 + r_1' \mathfrak{R}_2 + r_2' \mathfrak{R}_1 \} e \sin (2t-x)$$

(3)

$$+ \frac{1}{(2-2m+c)} \{ 2 r_4' + 2 r_0' \mathfrak{R}_4 + 2 r_4' \mathfrak{R}_0 + r_1' \mathfrak{R}_2 + r_2' \mathfrak{R}_1 \} e \sin (2t+x)$$

[4]

$$+ \frac{1}{m} \{ 2 r_5' + 2 r_0' \mathfrak{R}_5 + 2 r_5' \mathfrak{R}_0 + r_1' \mathfrak{R}_7 + r_1' \mathfrak{R}_6 + r_6' \mathfrak{R}_1 + r_7' \mathfrak{R}_1 \} e_i \sin z$$

[5]

$$+ \frac{1}{(2-3m)} \{ 2 r_6' + 2 r_0' \mathfrak{R}_6 + 2 r_6' \mathfrak{R}_0 + r_1' \mathfrak{R}_5 + r_5' \mathfrak{R}_1 \} e_i \sin (2t-z)$$

[6]

$$+ \frac{1}{(2-m)} \{ 2 r_7' + 2 r_0' \mathfrak{R}_7 + 2 r_7' \mathfrak{R}_0 + r_1' \mathfrak{R}_5 + r_5' \mathfrak{R}_1 \} e_i \sin (2t+z)$$

[7]

$$+ \frac{1}{2c} \{ 2 r_8' + 2 r_0' \mathfrak{R}_8 + 2 r_8' \mathfrak{R}_0 + r_1' \mathfrak{R}_{10} + r_1' \mathfrak{R}_9 + r_2' \mathfrak{R}_2 + r_3' \mathfrak{R}_4 + r_4' \mathfrak{R}_3 + r_9' \mathfrak{R}_1 + r_{10}' \mathfrak{R}_1 \} e^2 \sin 2x$$

[8]

$$+ \frac{1}{(2-2m-2c)} \{ 2 r_9' + 2 r_0' \mathfrak{R}_9 + 2 r_9' \mathfrak{R}_0 + r_1' \mathfrak{R}_8 + r_2' \mathfrak{R}_3 + r_3' \mathfrak{R}_2 + r_8' \mathfrak{R}_1 \} e^2 \sin (2t-2x)$$

[9]

$$+ \frac{1}{(2-2m+2c)} \{2r_{10}' + 2r_0 \mathfrak{R}_{10} + 2r_{10}' \mathfrak{R}_0 + r_1' \mathfrak{R}_8 + r_2' \mathfrak{R}_4 + r_4' \mathfrak{R}_2 + r_8' \mathfrak{R}_1\} e^2 \sin(2t+2z) \quad [10]$$

$$+ \frac{1}{(c+m)} \{2r_{11}' + 2r_0 \mathfrak{R}_{11} + 2r_{11}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{13} + r_1' \mathfrak{R}_{12} + r_2' \mathfrak{R}_5 + r_3' \mathfrak{R}_7 + r_4' \mathfrak{R}_6 + r_5' \mathfrak{R}_2 \\ + r_6' \mathfrak{R}_4 + r_7' \mathfrak{R}_3 + r_2' \mathfrak{R}_1 + r_{13}' \mathfrak{R}_1\} e e_j \sin(x+z) \quad [11]$$

$$+ \frac{1}{(2-3m-c)} \{2r_{12}' + 2r_0 \mathfrak{R}_{12} + 2r_{12}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{11} + r_2' \mathfrak{R}_6 + r_3' \mathfrak{R}_5 + r_5' \mathfrak{R}_3 + r_6' \mathfrak{R}_2 \\ + r_{11}' \mathfrak{R}_1\} e e_j \sin(2t-x-z) \quad [12]$$

$$+ \frac{1}{(2-m+c)} \{2r_{13}' + 2r_0 \mathfrak{R}_{13} + 2r_{13}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{11} + r_2' \mathfrak{R}_7 + r_4' \mathfrak{R}_5 + r_5' \mathfrak{R}_4 \\ + r_7' \mathfrak{R}_2 + r_{11}' \mathfrak{R}_1\} e e_j \sin(2t+x+z) \quad [13]$$

$$+ \frac{1}{(c-m)} \{2r_{14}' + 2r_0 \mathfrak{R}_{14} + 2r_{14}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{16} + r_1' \mathfrak{R}_{15} + r_2' \mathfrak{R}_5 + r_3' \mathfrak{R}_6 + r_4' \mathfrak{R}_7 + r_5' \mathfrak{R}_2 \\ + r_6' \mathfrak{R}_3 + r_7' \mathfrak{R}_4 + r_{15}' \mathfrak{R}_1 + r_{16}' \mathfrak{R}_1\} e e_j \sin(x-z) \quad [14]$$

$$+ \frac{1}{(2-m-c)} \{2r_{15}' + 2r_0 \mathfrak{R}_{15} + 2r_{15}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{14} + r_2' \mathfrak{R}_7 + r_3' \mathfrak{R}_5 + r_5' \mathfrak{R}_3 + r_7' \mathfrak{R}_2 \\ + r_{14}' \mathfrak{R}_1\} e e_j \sin(2t-x+z) \quad [15]$$

$$+ \frac{1}{(2-3m+c)} \{2r_{16}' + 2r_0 \mathfrak{R}_{16} + 2r_{16}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{14} + r_2' \mathfrak{R}_6 + r_4' \mathfrak{R}_5 + r_5' \mathfrak{R}_4 + r_6' \mathfrak{R}_2 \\ + r_{14}' \mathfrak{R}_1\} e e_j \sin(2t+x+z) \quad [16]$$

$$+ \frac{1}{2m} \{2r_{17}' + 2r_0 \mathfrak{R}_{17} + 2r_{17}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{19} + r_1' \mathfrak{R}_{18} + r_5' \mathfrak{R}_5 + r_6' \mathfrak{R}_7 + r_7' \mathfrak{R}_6 + r_{18}' \mathfrak{R}_3 \\ + r_{19}' \mathfrak{R}_1\} e_j^2 \sin 2z \quad [17]$$

$$+ \frac{1}{(2-4m)} \{2r_{18}' + 2r_0 \mathfrak{R}_{18} + 2r_{18}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{17} + r_5' \mathfrak{R}_6 + r_6' \mathfrak{R}_5 + r_{17}' \mathfrak{R}_1\} e_j^2 \sin(2t-2z) \quad [18]$$

$$+ \frac{1}{2} \{2r_{19}' + 2r_0 \mathfrak{R}_{19} + 2r_{19}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{17} + r_5' \mathfrak{R}_7 + r_7' \mathfrak{R}_5 + r_{17}' \mathfrak{R}_1\} e_j^2 \sin(2t+2z) \quad [19]$$

$$+ \frac{1}{1-m} \{2r_{101}' + 2r_0 \mathfrak{R}_{101} + 2r_{101}' \mathfrak{R}_0 + r_1' \mathfrak{R}_{101} + e^2 r_2' \mathfrak{R}_{102} + e^2 r_2' \mathfrak{R}_{103} + e^2 r_3' \mathfrak{R}_{102} \\ + e^2 r_4' \mathfrak{R}_{103} + e_j^2 r_5' \mathfrak{R}_{104} + e_j^2 r_5' \mathfrak{R}_{105} + r_{101}' \mathfrak{R}_1\} \sin t \quad [101]$$

These examples will serve for the present to show how the development may be obtained from Table II.

M. DAMOISEAU has given (Mém. sur la Théorie de la Lune, p. 348,) the expression for  $a \delta \frac{1}{r}$  in terms of the true longitude. In order to obtain a comparison of his results with those which may be obtained by the preceding method, it is necessary to transform his expressions, which may be done by LAGRANGE'S theorem, into series containing explicitly the mean longitude.

If we suppose

$$\frac{a}{r} = A_0 + A_1 \cos (2\lambda' - 2m\lambda') + e A_2 \cos (c\lambda' - \varpi) + e A_3 \cos (2\lambda' - 2m\lambda' - c\lambda' + \varpi) + \&c.$$

$$s = B_{146} \gamma \sin (g\lambda' - \nu) + B_{147} \gamma \sin (2\lambda' - 2m\lambda - g\lambda' + \nu) + B_{148} \gamma \sin (2\lambda' - 2m\lambda' + g\lambda' - \nu) + \&c.$$

$$nt = \lambda' + C_1 \sin (2\lambda' - 2m\lambda') + e C_2 \sin (c\lambda' - \varpi) + e C_3 \sin (2\lambda' - 2m\lambda' - c\lambda' + \varpi) + \&c.$$

in which expressions  $A$ ,  $B$ ,  $C$  are the same quantities as in M. DAMOISEAU'S notation, the indices only being changed according to the remark, Phil. Trans. 1830, p. 246, in order that Table II. may be applicable to the transformation required;  $\lambda'$  is called  $\nu$ , and  $\delta \cdot \frac{1}{r}$ ,  $\delta u$  in the notation of M. DAMOISEAU.

$$\frac{a}{r} = A_0 + \frac{1}{2} (2-2m) A_1 C_1 + \frac{c}{2} e^2 A_2 C_2 + \frac{1}{2} (2-2m-c) e^2 A_3 C_3 + \frac{1}{2} (2-2m+c) e^2 A_4 C_4$$

$$+ \frac{m}{2} e_1^2 A_5 C_5 + \&c.$$

$$+ \left\{ A_1 - \frac{1}{2} c e^2 A_2 C_3 + \frac{1}{2} c e^2 A_2 C_4 - \frac{1}{2} (2-2m-c) e^2 A_3 C_2 + \frac{1}{2} (2-2m+c) e^2 A_4 C_2 \right. \\ \left. - \frac{1}{2} m e_1^2 A_5 C_6 + \frac{1}{2} m e_1^2 A_5 C_7 - \frac{1}{2} (2-3m) e_1^2 A_6 C_5 + \frac{1}{2} (2-m) e_1^2 A_7 C_5 \right\} \cos 2t$$

[1]

$$+ \left\{ A_2 + \frac{1}{2} (2-2m) A_1 C_4 + \frac{1}{2} (2-2m) A_1 C_3 + \frac{1}{2} (2-2m-c) A_3 C_1 \right.$$

$$\left. + \frac{1}{2} (2-2m+c) A_4 C_1 \right\} e \cos x$$

[2]

$$+ \left\{ A_3 + \frac{1}{2} (2-2m) A_1 C_2 + \frac{c}{2} A_2 C_1 \right\} e \cos (2t-x)$$

[3]

$$+ \left\{ A_4 - \frac{1}{2} (2-2m) A_1 C_2 - \frac{c}{2} A_2 C_1 \right\} e \cos (2t+x)$$

[4]

$$+ \left\{ A_5 + \frac{1}{2} (2-2m) A_1 C_7 + \frac{1}{2} (2-2m) A_1 C_6 + \frac{1}{2} (2-3m) A_6 C_1 \right. \\ \left. + \frac{1}{2} (2-m) A_7 C_1 \right\} e_i \cos z$$

[5]

$$+ \left\{ A_6 + \frac{1}{2} (2-2m) A_1 C_5 + \frac{m}{2} A_5 C_1 \right\} e_i \cos (2t-z)$$

[6]

$$+ \left\{ A_7 - \frac{1}{2} (2-2m) A_1 C_5 - \frac{m}{2} A_5 C_1 \right\} e_i \cos (2t+z)$$

[7]

$$+ \left\{ A_8 + \frac{1}{2} (2-2m) A_1 C_{10} + \frac{1}{2} (2-2m) A_1 C_9 - \frac{c}{2} A_2 C_2 + \frac{1}{2} (2-2m-c) A_3 C_4 \right. \\ \left. + \frac{1}{2} (2-2m+c) A_4 C_3 + \frac{1}{2} (2-2m-2c) A_9 C_1 + \frac{1}{2} (2-2m+2c) A_{10} C_1 \right\} e^2 \cos 2x$$

[8]

$$+ \left\{ A_9 + \frac{1}{2} (2-2m) A_1 C_8 + \frac{c}{2} A_2 C_3 + \frac{1}{2} (2-2m-c) A_3 C_2 + c A_8 C_1 \right\} e^2 \cos (2t-2x)$$

[9]

$$+ \left\{ A_{10} - \frac{1}{2} (2-2m) A_1 C_8 - \frac{c}{2} A_2 C_4 - \frac{1}{2} (2-2m+c) A_4 C_2 - c A_8 C_1 \right\} e^2 \cos (2t+2x)$$

[10]

$$+ \left\{ A_{11} + \frac{1}{2} (2-2m) A_1 C_{13} + \frac{1}{2} (2-2m) A_1 C_{12} - \frac{c}{2} A_2 C_5 + \frac{1}{2} (2-2m-c) A_3 C_7 \right. \\ \left. + \frac{1}{2} (2-2m+c) A_4 C_6 - \frac{m}{2} A_5 C_2 + \frac{1}{2} (2-3m) A_6 C_4 + \frac{1}{2} (2-m) A_7 C_3 \right. \\ \left. + \frac{1}{2} (2-3m-c) A_{12} C_1 \right\} e e_i \cos (x+z)$$

[11]

$$+ \left\{ A_{12} + \frac{1}{2} (2-2m) A_1 C_{11} + \frac{c}{2} A_2 C_6 + \frac{1}{2} (2-2m-c) A_3 C_5 + \frac{m}{2} A_5 C_3 \right. \\ \left. + \frac{1}{2} (2-3m) A_6 C_2 + \frac{1}{2} (c+m) A_{11} C_1 \right\} e e_i \cos (2t-x-z)$$

[12]

$$+ \left\{ A_{13} - \frac{1}{2} (2-2m) A_1 C_{11} - \frac{c}{2} A_2 C_7 - \frac{1}{2} (2-2m+c) A_4 C_5 - \frac{m}{2} A_5 C_4 \right. \\ \left. - \frac{1}{2} (2-m) A_7 C_2 - \frac{1}{2} (c+m) A_{11} C_1 \right\} e e_i \cos (2t+x+z)$$

[13]

$$+ \left\{ A_{14} + \frac{1}{2} (2-2m) A_1 C_{16} + \frac{1}{2} (2-2m) A_1 C_{15} + \frac{c}{2} A_2 C_5 + \frac{1}{2} (2-2m-c) A_3 C_6 \right. \\ \left. + \frac{1}{2} (2-2m+c) A_4 C_7 + \frac{m}{2} A_5 C_2 + \frac{1}{2} (2-3m) A_6 C_3 + \frac{1}{2} (2-m) A_7 C_4 \right.$$

$$+ \frac{1}{2} (2 - 3m + c) A_{15} C_1 + \frac{1}{2} (2 - m - c) A_{16} C_1 \} e e_i \cos (x - z) \quad [14]$$

$$+ \left\{ A_{15} + \frac{1}{2} (2 - 2m) A_1 C_{14} + \frac{c}{2} A_2 C_7 - \frac{1}{2} (2 - 2m - c) A_3 C_5 - \frac{m}{2} A_5 C_3 \right. \\ \left. + \frac{1}{2} (2 - m) A_7 C_2 + \frac{1}{2} (c - m) A_{14} C_1 \right\} e e_i \cos (2t - x + z) \quad [15]$$

$$+ \left\{ A_{16} - \frac{1}{2} (2 - 2m) A_1 C_{14} - \frac{c}{2} A_2 C_6 + \frac{1}{2} (2 - 2m + c) A_4 C_5 + \frac{m}{2} A_5 C_4 \right. \\ \left. - \frac{1}{2} (2 - 3m) A_6 C_2 - \frac{1}{2} (c - m) A_{14} C_1 \right\} e e_i \cos (2t + x - z) \quad [16]$$

$$+ \left\{ A_{17} + \frac{1}{2} (2 - 2m) A_1 C_{19} + \frac{1}{2} (2 - 2m) A_1 C_{18} - \frac{m}{2} A_5 C_5 + \frac{1}{2} (2 - 3m) A_6 C_7 \right. \\ \left. + \frac{1}{2} (2 - m) A_7 C_6 + \frac{1}{2} (2 - 4m) A_{18} C_1 + A_{19} C_1 \right\} e_i^2 \cos 2z \quad [17]$$

$$+ \left\{ A_{18} + \frac{1}{2} (2 - 2m) A_1 C_{17} + \frac{m}{2} A_5 C_6 + \frac{1}{2} (2 - 3m) A_6 C_5 + m A_{17} C_1 \right\} e_i^2 \cos (2t - 2z) \quad [18]$$

$$+ \left\{ A_{19} - \frac{1}{2} (2 - 2m) A_1 C_{17} - \frac{m}{2} A_5 C_7 - \frac{1}{2} (2 - m) A_6 C_5 - m A_{17} C_1 \right\} e_i^2 \cos (2t + 2z) \quad [19]$$

Similarly

$$s = \left\{ B_{146} + \frac{1}{2} (2 - 2m + g) C_1 B_{148} - \frac{1}{2} (2 - 2m - g) C_1 B_{147} + \frac{1}{2} (c + g) e^2 C_2 B_{150} \right. \\ - \frac{1}{2} (c - g) e^2 C_2 B_{149} + \frac{1}{2} (2 - 2m - c + g) e^2 C_3 B_{152} - \frac{1}{2} (2 - 2m - c - g) e^2 C_3 B_{151} \\ + \frac{1}{2} (2 - 2m + c + g) C_4 B_{154} - \frac{1}{2} (2 - 2m + c - g) C_4 B_{153} \\ \left. + \frac{1}{2} (m + g) e_i^2 C_5 B_{156} - \frac{1}{2} (m - g) e_i^2 C_5 B_{155} \right\} \gamma \sin y \quad [146]$$

$$+ \left\{ B_{147} - \frac{g}{2} C_1 B_{146} - \frac{1}{2} (2 - 2m - c - g) e^2 C_2 B_{151} + \frac{1}{2} (2 - 2m + c - g) e^2 C_2 B_{153} \right. \\ - \frac{1}{2} (c - g) e^2 C_3 B_{149} - \frac{1}{2} (c + g) C_4 B_{150} - \frac{1}{2} (2 - 3m - g) e_i^2 C_5 B_{157} \\ \left. + \frac{1}{2} (2 - m - g) e_i^2 C_5 B_{159} \right\} \gamma \sin (2t - y) \quad [147]$$

$$\begin{aligned}
& + \left\{ B_{148} - \frac{g}{2} C_1 B_{146} - \frac{1}{2} (2 - 2m - c + g) e^2 C_2 B_{152} + \frac{1}{2} (2 - 2m + c + g) e^2 C_2 B_{154} \right. \\
& \quad - \frac{1}{2} (c + g) e^2 C_3 B_{150} - \frac{1}{2} (c - g) C_4 B_{149} - \frac{1}{2} (2 - 3m + g) e_l^2 C_5 B_{158} \\
& \quad \left. + \frac{1}{2} (2 - m + g) e_l^2 C_5 B_{160} \right\} \gamma \sin (2t + y) \\
& \qquad \qquad \qquad [148]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{149} + \frac{1}{2} (2 - 2m + c - g) C_1 B_{153} - \frac{1}{2} (2 - 2m - c + g) C_1 B_{152} - \frac{g}{2} C_2 B_{146} \right. \\
& \quad \left. + \frac{1}{2} (2 - 2m - g) C_3 B_{147} - \frac{1}{2} (2 - 2m + g) C_4 B_{148} \right\} e \gamma \sin (x - y) \\
& \qquad \qquad \qquad [149]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{150} + \frac{1}{2} (2 - 2m + c + g) C_1 B_{154} - \frac{1}{2} (2 - 2m - c - g) C_1 B_{151} - \frac{g}{2} C_2 B_{146} \right. \\
& \quad \left. + \frac{1}{2} (2 - 2m + g) C_3 B_{148} - \frac{1}{2} (2 - 2m - g) C_4 B_{147} \right\} e \gamma \sin (x + y) \\
& \qquad \qquad \qquad [150]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{151} - \frac{1}{2} (c + g) C_1 B_{150} + \frac{1}{2} (2 - 2m - g) C_2 B_{147} - \frac{g}{2} C_3 B_{146} \right\} e \gamma \sin (2t - x - y) \\
& \qquad \qquad \qquad [151]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{152} - \frac{1}{2} (c - g) C_1 B_{149} + \frac{1}{2} (2 - 2m + g) C_2 B_{148} - \frac{g}{2} C_3 B_{146} \right\} e \gamma \sin (2t - x + y) \\
& \qquad \qquad \qquad [152]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{153} - \frac{1}{2} (c - g) C_1 B_{149} - \frac{1}{2} (2 - 2m - g) C_2 B_{147} - \frac{g}{2} C_4 B_{146} \right\} e \gamma \sin (2t + x - y) \\
& \qquad \qquad \qquad [153]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{154} - \frac{1}{2} (c + g) C_1 B_{150} - \frac{1}{2} (2 - 2m + g) C_2 B_{148} - \frac{g}{2} C_4 B_{146} \right\} e \gamma \sin (2t + x + y) \\
& \qquad \qquad \qquad [154]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{155} + \frac{1}{2} (2 - m - g) C_1 B_{159} - \frac{1}{2} (2 - 3m + g) C_1 B_{158} - \frac{g}{2} C_5 B_{146} \right\} e_l \gamma \sin (z - y) \\
& \qquad \qquad \qquad [155]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{156} + \frac{1}{2} (2 - m + g) C_1 B_{160} - \frac{1}{2} (2 - 3m - g) C_1 B_{157} - \frac{g}{2} C_5 B_{146} \right\} e_l \gamma \sin (z + y) \\
& \qquad \qquad \qquad [156]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{157} - \frac{1}{2} (m + g) C_1 B_{156} + \frac{1}{2} (2 - 2m - g) C_5 B_{147} \right\} e_l \gamma \sin (2t - z - y) \\
& \qquad \qquad \qquad [157]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{158} - \frac{1}{2} (m - g) C_1 B_{155} + \frac{1}{2} (2 - 2m + g) C_5 B_{148} \right\} e_l \gamma \sin (2t - z + y) \\
& \qquad \qquad \qquad [158]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ B_{159} - \frac{1}{2} (m - g) C_1 B_{155} - \frac{1}{2} (2 - 2m - g) C_5 B_{147} \right\} e_l \gamma \sin (2t + z - y) \\
& \qquad \qquad \qquad [159]
\end{aligned}$$



$$+ \left\{ B_{160} - \frac{1}{2} (m + g) C_1 B_{156} - \frac{1}{2} (2 - 2m + g) C_5 B_{148} \right\} e_i \gamma \sin (2t + z + y) \quad [160]$$

In order to verify these expressions, suppose

$$\frac{a}{r} = A_2 e \cos (c \lambda' - \varpi) \quad s = \gamma B_{146} \sin (g \lambda' - \nu) \quad nt = \lambda' + C_1 \sin (2 \lambda' - 2m \lambda')$$

Then by LAGRANGE's theorem, neglecting  $A^3$ ,  $A^2 C$ , &c.

$$\frac{a}{r} = A_2 e \cos x + c e A_2 C_1 \sin 2t \sin x \quad \text{nearly}$$

$$= A_2 e \cos x + \frac{c A_2 C_1}{2} e \cos (2t - x) - \frac{c A_2 C_1}{2} e \cos (2t + x) \quad [2] \quad [3] \quad [4]$$

which terms are found in the expression which I have given above.

Again, by LAGRANGE's theorem,

$$\begin{aligned} s &= \gamma B_{146} \sin y - g \gamma C_1 B_{146} \sin 2t \cos y \\ &= \gamma B_{146} \sin y - \frac{g C_1 B_{146}}{2} \gamma \sin (2t - y) - \frac{g C_1 B_{146}}{2} \gamma \sin (2t + y) \end{aligned} \quad [146] \quad [147] \quad [148]$$

which terms are found in the expression which I have given above.

The numerical values of the quantities  $A$ ,  $B$ ,  $C$ , according to M. DAMOISEAU, are

$A_0 = ?$	[30] $A_1 = \cdot 00709538$	[1] $A_2 = ?$
* [31] $A_3 = \cdot 2024622$	[32] $A_4 = - \cdot 00369361$	[16] $A_5 = - \cdot 0056375$
[33] $A_6 = \cdot 0289158$	[34] $A_7 = - \cdot 0030859$	[2] $A_8 = \cdot 003183 ?$
[35] $A_9 = \cdot 347942$	[36] $A_{10} = \cdot 001970$	[19] $A_{11} = - \cdot 19737$
[41] $A_{12} = \cdot 516174$	[42] $A_{13} = \cdot 0026238$	[18] $A_{14} = - \cdot 286046$
[39] $A_{15} = - \cdot 060625$	[40] $A_{16} = - \cdot 014546$	[17] $A_{17} = - \cdot 006930$
[43] $A_{18} = \cdot 08125$		
[30] $C_1 = - \cdot 009216$	[1] $C_2 = - \cdot 20044055$	[31] $C_3 = - \cdot 4138664$
[32] $C_4 = \cdot 012939$	[16] $C_5 = - \cdot 194385$	[33] $C_6 = - \cdot 394172$
[34] $C_7 = \cdot 0038267$	[2] $C_8 = \cdot 745169$	[35] $C_9 = - \cdot 286413$
[36] $C_{10} = - \cdot 012575$	[19] $C_{11} = \cdot 365516$	[41] $C_{12} = - \cdot 108891$
[42] $C_{13} = - \cdot 008551$	[18] $C_{14} = - \cdot 607534$	[39] $C_{15} = \cdot 11587$
[40] $C_{16} = \cdot 055936$	[17] $C_{17} = \cdot 12755$	[43] $C_{18} = - \cdot 11432$

\* These are the indices of the arguments in M. DAMOISEAU's work.

$$\begin{array}{lll}
[0] \quad B_{147} = \cdot 0284942 & [2] \quad B_{149} = -\cdot 019169 & [6] \quad B_{151} = -\cdot 020788 \\
[5] \quad B_{153} = \cdot 006113 & [8] \quad B_{155} = -\cdot 081170 & [11] \quad B_{157} = \cdot 071237 \\
[10] \quad B_{159} = -\cdot 0033394 & & 
\end{array}$$

Having found the coefficients of  $\frac{a}{r}$ , those of  $\frac{a}{r}$  are easily determined.

$$\begin{aligned}
\frac{a}{r} &= \frac{a}{r(1+s^2)} = \frac{a}{r} \left\{ 1 - \frac{s^2}{2} \right\} \\
&= \frac{a}{r} \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 + \frac{\gamma^2}{2} s_{147} \cos 2t \right. \\
&\quad \left. + \frac{\gamma^2}{4} \cos 2y - \frac{\gamma^2}{2} s_{147} \cos (2t - 2y) \right\}
\end{aligned}$$

If the coefficients of  $\frac{a}{r}$  be called  $r_n$ ,

$$\begin{aligned}
r_0 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_0 + \frac{\gamma^2}{4} s_{147} r_1 \\
r_1 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_1 \\
r_2 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_2 + \frac{\gamma^2}{4} s_{147} r_3 + \frac{\gamma^2}{4} s_{147} r_4 \\
r_3 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_3 + \left( 1 - \frac{e^2}{8} \right) \frac{\gamma^2}{4} s_{147} \\
r_4 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_4 + \left( 1 - \frac{e^2}{8} \right) \frac{\gamma^2}{4} s_{147} \\
r_5 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_5 \\
r_6 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_6 + \frac{\gamma^2}{4} s_{147} r_5 \\
r_7 &= \left\{ 1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s_{147}^2 \right\} r_7 + \frac{\gamma^2}{4} s_{147} r_6
\end{aligned}$$

If we suppose

$$\frac{a}{r} = 1 + r_0 + e(1+f) \cos \left( n(1+k)t + \varepsilon - \varpi \right) + e_i f_i \cos \left( n(1+k_i)t + \varepsilon_i - \varpi_i \right)$$

$a < a_i$  we find

$$\begin{aligned}
r_0 &= \frac{m_i}{\mu} \left\{ \frac{a^3}{2a_i^3} b_{3,0} - \frac{a^2}{2a_i^2} b_{3,1} \right\} & k &= \frac{m_i}{\mu} \left\{ \frac{a^3}{a_i^3} b_{3,0} - \frac{5a^2}{4a_i^2} b_{3,1} \right\} \\
f_i \{ (1+k_i)^2 (1-3r_0) - 1 \} &= \frac{m_i a^2}{2\mu a_i^2} b_{3,2}
\end{aligned}$$

$$\text{If } n\{1+2r_0\}=n \text{ and } n^2=\frac{\mu}{a^3} \quad a=a\left\{1+\frac{4}{3}r_0\right\}$$

If  $2e$  is the coefficient of  $\sin(n(1+k)t+\varepsilon-\varpi)$  in the expression for the longitude,

$$e(1+f)=e(1+k-r_0)$$

$$\begin{aligned} \frac{a}{r} &= 1 - \frac{1}{3}r_0 + e\left\{1+k-\frac{7}{3}r_0\right\}\cos\left(n(1+k)t+\varepsilon-\varpi\right) \\ &\quad + e_1f_1\cos\left(n(1+k_1)t+\varepsilon-\varpi_1\right) \\ &= 1 - \frac{m_1a^3}{6\mu a_1^3}b_{3,0} + \frac{m_1a^2}{6\mu a_1^2}b_{3,1} \\ &\quad + e\left\{1 - \frac{m_1a^3}{6\mu a_1^3}b_{3,0} - \frac{m_1a^2}{12\mu a_1^2}b_{3,1}\right\}\cos\left(n\left(1 - \frac{m_1a^2}{4\mu a_1^2}b_{3,1}\right)t+\varepsilon-\varpi\right) \\ &\quad + e_1f_1\cos\left(n(1+k_1)t+\varepsilon-\varpi_1\right) \\ \frac{r}{a} &= 1 + \frac{1}{3}r_0 - e\left\{1+k-\frac{5}{3}r_0\right\}\cos\left(n(1+k)t+\varepsilon-\varpi\right) \\ &\quad - e_1f_1\cos\left(n(1+k_1)t+\varepsilon-\varpi_1\right) \\ &= 1 + \frac{m_1a^3}{6\mu a_1^3}b_{3,0} - \frac{m_1a^2}{6\mu a_1^2}b_{3,1} \\ &\quad - e\left\{1 + \frac{m_1a^3}{6\mu a_1^3}b_{3,0} - \frac{5m_1a^2}{12\mu a_1^2}b_{3,1}\right\}\cos\left(n\left(1 - \frac{m_1a^2}{4\mu a_1^2}b_{3,1}\right)t+\varepsilon-\varpi\right) \\ &\quad - e_1f_1\cos\left(n(1+k_1)t+\varepsilon-\varpi_1\right) \end{aligned}$$

If  $a < a_1$  as before, and

$$\frac{a_1}{r_1} = 1 + r_{10} + e_1(1+f_1)\cos\left(n_1(1+k_1)t+\varepsilon_1-\varpi_1\right) + e_1f_1'\cos\left(n_1(1+k_1')t+\varepsilon_1-\varpi_1\right)$$

we find

$$r_{10} = \frac{m}{\mu}\left\{\frac{1}{2}b_{3,0} - \frac{a}{2a_1}b_{3,1}\right\} \quad k_1 = \frac{m}{\mu_1}\left\{b_{3,0} - \frac{5a}{4a_1}b_{3,1}\right\}$$

$$f_1'\left\{(1+k_1')^2(1-3r_{10})-1\right\} = \frac{ma}{2\mu_1a_1}b_{3,2}$$

$$\text{If } n_1\{1+2r_0\}=n_1 \text{ and } n_1^2=\frac{\mu}{a_1^3}, \quad a_1=a_1\left\{1+\frac{4}{3}r_{10}\right\}$$

$$\begin{aligned} \frac{a_i}{r_i} &= 1 - \frac{m}{6\mu_i} b_{3,0} + \frac{ma}{6\mu_i a_i} b_{3,1} \\ &+ e_i \left\{ 1 + \frac{m}{6\mu_i} b_{3,0} - \frac{ma}{12\mu_i a_i} b_{3,1} \right\} \cos \left( n_i \left( 1 - \frac{ma}{4\mu_i a_i} b_{3,1} \right) t + \varepsilon_i - \varpi_i \right) \\ &+ e f'_i \cos \left( n_i (1 + k'_i) t + \varepsilon_i - \varpi \right) \end{aligned}$$

$\mu$  is the mass of the sun + the mass of the disturbed planet, which is not of course the same for both, but the difference may be neglected in the planetary theory.

LAPLACE determines the arbitrary quantity  $f_i$ , upon the hypothesis that the coefficient of the argument  $\sin \left( n (1 + k) t + \varepsilon - \varpi_i \right)$  in the expression for the longitude equals zero. According to the received theory of the moon, the true longitude is expressed in a series of angles consisting of various combinations of the quantities  $t, x, y$  and  $z$ , and their multiples and no others; and in this theory the angle  $t + z$  occupies the place of the argument  $nt + \varepsilon - \varpi_i$ , so that omitting  $\varepsilon$  which accompanies  $t$ ,

$$\frac{a}{r} = 1 + r_0 + e (1 + f) \cos (c n t - \varpi) + e_i f_i \cos (n t - n_i t + c_i n_i t - \varpi_i)$$

$$\frac{a_i}{r_i} = 1 + r_{i,0} + e_i (1 + f'_i) \cos (c' n_i t - \varpi_i) + e f'_i \cos (n_i t - n t + c n t - \varpi)$$

$$c^* = 1 - \frac{m_i a^2}{4\mu a_i^2} b_{3,1} \quad c_i = 1 - \frac{ma}{4\mu_i a_i} b_{3,1} = 1 \text{ nearly}$$

$$n_i (c_i - 1) = n k_i = 0 \text{ nearly}$$

$$f_i \left\{ (1 + k_i)^2 (1 - 3 r_0) - 1 \right\} = \frac{m_i a^2}{2\mu a_i^2} b_{3,2} = \frac{15 m_i a^4}{8\mu a_i^4}$$

$$r_0 = -\frac{m_i a^3}{2\mu a_i^3} \quad f_i = \frac{5a}{4a_i}$$

\*  $c$  and  $g$  are determined by quadratic equations,

$$c = \frac{\sqrt{\left\{ 1 + \frac{m_i}{\mu} \left\{ \frac{a^3}{2a_i^3} b_{3,0} - \frac{a^2}{a_i^2} b_{3,1} \right\} \right\}}}{1 + \frac{m_i}{\mu} \left\{ \frac{a^3}{2a_i^3} b_{3,0} - \frac{a^2}{2a_i^2} b_{3,1} \right\}} = 1 - \frac{m_i a^2}{4\mu a_i^2} b_{3,1} \text{ nearly.}$$

This gives for the coefficient of  $\sin(t + z)$  in the expression for the longitude

$$+ \left\{ \frac{5a}{2a_i} - \frac{3m_i a^4}{8\mu a_i^4} \right\} e_i$$

which in sexagesimal seconds is  $21''.7$ , according to M. DAMOISEAU it should be  $17''.56$ .

Finally,

$$\frac{a}{r} = 1 + \frac{m_i a^3}{6\mu a_i^3} + e \left\{ 1 - \frac{7m_i a^3}{12\mu a_i^3} \right\} \cos x + \frac{5a}{4a_i} e_i \cos(t + z)$$

$$\lambda = nt + 2e \sin x + \left\{ \frac{5a}{2a_i} - \frac{3m_i a^4}{8\mu a_i^4} \right\} e_i \sin(t + z)$$

Substituting for  $b_{3,1}$ ,  $b_{3,2}$  their values in series

$$b_{3,1} = \frac{3a}{a_i} + \frac{3 \cdot 3 \cdot 5 a^3}{2 \cdot 4 a_i^3} + \&c. \quad b_{3,2} = \frac{3 \cdot 5 a^2}{4 a_i^2} + \frac{3 \cdot 3 \cdot 5 \cdot 7 a^4}{2 \cdot 4 \cdot 6 a_i^4} + \&c.$$

$$c = 1 - \frac{3m_i a^3}{4\mu a_i^3} \quad c_i = 1 - \frac{3ma^2}{4\mu a_i^2}$$

I have shown, Phil. Trans. 1832, p. 38, that when  $a < a_i$

$$g = 1 + \frac{m_i}{\mu} \left\{ \frac{a^3}{a_i^3} b_{3,0} - \frac{3a^2}{4a_i^3} b_{3,1} \right\}$$

$$gn = n \left\{ 1 + \frac{m_i a^3}{4\mu a_i^3} b_{3,1} \right\}$$

Similarly it may be shown that

$$g_i = 1 + \frac{m}{\mu_i} \left\{ b_{3,0} - \frac{3a}{4a_i} b_{3,1} \right\}$$

$$g_i n_i = n_i \left\{ 1 + \frac{ma}{4\mu_i a_i} b_{3,1} \right\}$$

The arguments

$$nt - \nu, nt - \nu_i, nt_i - \nu_i \text{ and } n_i t - \nu$$

occupy the same place in the expression for the latitude as

$$nt - \varpi, nt - \varpi_i, n_i t - \varpi_i \text{ and } n_i t - \varpi$$

in the expression for the radius vector. Similar methods may be employed to determine the arbitrary quantities, so that no other angles occur in the expression for  $s$  except the quantities  $t, x, z, y$ , and if the quantities  $c$  and  $g$  are rational, no imaginary angles can be introduced.